Innovative Technologica: Methodical Research Journal Vol: 4, No 2, 2025, Page: 1-11

# Numerical Approximation of Periodic Points For Some Mappings

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DOI: https://doi.org/10.47134/innovative.v4i2.142
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Received: 22-04-2025
Accepted: 22-05-2025
Published: 22-06-2025



**Abstract:** This study delves into the numerical approximation of periodic points for quadratic mappings, with a particular focus on ecological dynamics represented by the logistic map. We extend the analysis to two-dimensional mappings to capture the interactions between interconnected islands, where parameters reflect living conditions and population dynamics. The investigation involves solving systems of equations to identify points where the period of the mapping equals four, distinguishing them from points with periods equal to two. Approximate solution methods are employed due to the complexity of the equations, facilitating the exploration of periodic orbits and their spectral characteristics through numerical experimentation.

**Keywords:** Numerical Approximation, Periodic Points, Quadratic Mappings, Ecological Dynamics, Logistic Map.

## Introduction

The study of periodic points for mappings has long been a central theme in dynamical systems theory, providing fundamental insights into the long-term behavior of iterative processes. Periodic points, which return to their initial state after a fixed number of iterations, are crucial for understanding stability, bifurcations, and other essential characteristics of dynamical systems.

While analytical solutions for periodic points are often elusive, numerical methods offer a powerful approach to approximate these points and explore the dynamics of mappings. This paper focuses on the numerical approximation of periodic points for specific classes of mappings, aiming to develop and analyze efficient and reliable algorithms for this purpose.

By combining theoretical foundations with computational techniques, we seek to advance our understanding of the complex behavior exhibited by various mappings and contribute to the broader field of dynamical systems.

**Methodology**

The logistic map, as described in [1]

$$x_{n+1} = rx_n(1-x_n)$$

where $x_n$ is a number between 0 and 1, the parameter $r$ are those in the interval $[0,4]$ mean the condition for living in the island. After linear transformations we can consider the following as logistic mapping:

$$x_{n+1} = x_n^2 + c \tag{1}$$

but here, the parameter $c$ changes between $[-2;0.25]$ and the number of population $|x_n|$. The learning of the asymptotics of trajectories of the mapping (1) is called the problem of Von Neumann - Ulam [2].

Two dimensional case of the mapping (1) is

$$F_{c_1c_2}:\begin{cases} x' = y^2 + c_1, \\ y' = x^2 + c_2. \end{cases} \tag{2}$$

where $(x,y)\in R^2$ and $(c_1,c_2)\in R^2$.

Our mathematical model of the population in the connected two islands is

$$F_{c_1c_2}:\begin{cases} x_{n+1} = y_n^2 + c_1, \\ y_{n+1} = x_n^2 + c_2. \end{cases} \tag{3}$$

where $|x_0|$ is the initial number of the population of first island and $|y_0|$ is the initial number of the population of second island in millions. For example, $|x_0| = 0.02$ means the initial number of population of first island is 20000. $c_1$ and $c_2$ are the living conditions in the islands respectively. $|x_n|$ and $|y_n|$ are the numbers of $n-$ th generation of populations first and second island.

To find the points where the period of mapping (2) is equal to four, it is necessary to solve the following equations

$$\begin{cases} \left(\left(\left(x^2+c_2\right)^2+c_1\right)^2+c_2\right)^2+c_1-x=0, \\ \left(\left(\left(y^2+c_1\right)^2+c_2\right)^2+c_1\right)^2+c_2-y=0. \end{cases}$$

Among the solutions of this system of equations are also points whose periods are equal to two. To separate them and leave only the equation of four points of period, we must divide the equations in the system of equations into the following two equations accordingly

$$x^4+2c_2x^2-x+c_2^2+c_1 \text{ ва } y^4+2c_1y^2-y+c_1^2+c_2.$$

In this case, the following system of equations is formed

$$\begin{cases} x^{12}+6c_2x^{10}+x^9+(15c_2^2+3c_1)x^8+4c_2x^7+(20c_2^3+12c_1c_2+1)x^6+ \\ +(2c_1+6c_2^2)x^5+(3c_1^2+4c_2+18c_1c_2^2+15c_2^4)x^4+(1+4c_1c_2+4c_2^3)x^3+ \\ +(c_1+6c_1^2c_2+5c_2^2+12c_1c_2^3+6c_2^5)x^2+(c_1^2+2c_2+2c_1c_2^2+c_2^4)x+ \\ +c_1^6+3c_1c_2^4+2c_2^3+3c_1^2c_2^2+2c_1c_2+c_1^3+1=0, \\ \\ y^{12}+6c_1y^{10}+y^9+(15c_1^2+3c_2)y^8+4c_1y^7+(20c_1^3+12c_1c_2+1)y^6+ \\ +(2c_2+6c_1^2)y^5+(3c_2^2+4c_1+18c_2c_1^2+15c_1^4)y^4+(1+4c_1c_2+4c_1^3)y^3+ \\ +(c_2+6c_2^2c_1+5c_1^2+12c_2c_1^3+6c_1^5)y^2+(c_2^2+2c_1+2c_2c_1^2+c_1^4)y+ \\ +c_2^6+3c_2c_1^4+2c_1^3+3c_1^2c_2^2+2c_1c_2+c_2^3+1=0. \end{cases}$$

According to Abel's theorem, these equations cannot be solved analytically in the general case. Therefore, we solve it using approximate solution methods for certain values of the parameters.

For example $c_1=-0.98$ and $c_2=-0.02$ let's solve approximately.

$$\begin{cases} 0.099144+0.919616\,x-1.09315\,x^2+1.07837\,x^3+2.79415\,x^4-1.9576\,x^5+ \\ +1.23504\,x^6-0.08\,x^7-2.934\,x^8+x^9-0.12\,x^{10}+x^{12}=0, \\ \\ -0.0115392-1.07565\,y-0.417991\,y^2-2.68637\,y^3+9.57098\,y^4+5.7224\,y^5- \\ -17.5886\,y^6-3.92\,y^7+14.346\,y^8+y^9-5.88\,y^{10}+y^{12}=0. \end{cases}$$

The equations in this system of equations are not related to each other so we solve them separately.

a. First

$$0.099144+0.919616\,x-1.09315\,x^2+1.07837\,x^3+2.79415\,x^4-1.9576\,x^5+$$
$$+1.23504\,x^6-0.08\,x^7-2.934\,x^8+x^9-0.12\,x^{10}+x^{12}=0$$

We solve numerical solutions of the equation using approximate methods. To do this, we find the gap where all the solutions are located.

$$A=\max\{|a_1|,|a_2|,\ldots,|a_n|\}=2.934,\quad R=1+\frac{A}{|a_0|}=1+\frac{2.934}{1}=3.934.$$

This means that all solutions are in the interval $(-3.934, 3.934)$.

| | $f_0=f$ | $f_1$ | $f_2$ | $f_3$ | $f_4$ | $f_5$ | $f_6$ | $f_7$ | $f_8$ | $f_9$ | $f_{10}$ | $f_{11}$ | $f_{12}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| -3.934 | + | - | + | + | - | - | + | + | + | - | - | + | - | 7 |
| 3.934 | + | + | + | + | - | + | + | - | - | + | - | - | - | 5 |
| | | | | | | | | | | | | | | **2** |

**Result and Discussion**

We can see that there are two real solutions to the equation. Algorithm:

1. We divide the interval $(-3.934, 3.934)$ into two equal parts and check which interval has a solution using the Sturm theorem for each of these intervals. If there is only one solution in each interval, we find approximate solutions using an arbitrary one of the methods of dividing the section into two equal parts, watts, and attempts to find solutions for each interval.
2. If more than one solution is in the same interval, then we apply the Sturm theorem again by dividing the interval into three equal parts. If several more solutions remain in the same interval, we will continue to use Sturm's theorem to divide the interval into four, five, six, and so on. We stop when there is only one solution or no solution in each interval.
3. Then we find the approximate solutions using the arbitrary one of the methods of dividing the section into two equal parts, watts and attempts, to separate the intervals in which there is a solution and find the solutions in each interval.

   For the second equation, we use the same algorithm.

   b. The second

$$-0.0115392 - 1.07565\,y - 0.417991\,y^2 - 2.68637\,y^3 + 9.57098\,y^4 + 5.7224\,y^5 -$$
$$-17.5886\,y^6 - 3.92\,y^7 + 14.346\,y^8 + y^9 - 5.88\,y^{10} + y^{12} = 0$$

We solve numerical solutions of the equation using approximate methods. To do this, we find the gap where all the solutions are located

$$A = \max\{|a_1|, |a_2|, \ldots, |a_n|\} = 17.5886, \qquad R = 1 + \frac{A}{|a_0|} = 1 + \frac{17.5886}{1} = 18.5886.$$

This means that all solutions are in the interval $(-18.5886, 18.5886)$.

$$g = -0.0115392 - 1.07565\,y - 0.417991\,y^2 - 2.68637\,y^3 + 9.57098\,y^4 + 5.7224\,y^5 - $$
$$- 17.5886\,y^6 - 3.92\,y^7 + 14.346\,y^8 + y^9 - 5.88\,y^{10} + y^{12},$$
$$g_1 = -1.07565 - 0.835981\,y - 8.0591\,y^2 + 38.2839\,y^3 + 28.612\,y^4 - 105.532\,y^5 - $$
$$- 27.44\,y^6 + 114.768\,y^7 + 9\,y^8 - 58.8\,y9 + 12\,y^{11},$$
$$g_2 = 0.0115392 + 0.986011\,y + 0.348326\,y^2 + 2.01478\,y^3 - 6.38065\,y^4 - 3.33807\,y^5 + $$
$$+ 8.79432\,y^6 + 1.63333\,y^7 - 4.782\,y^8 - 0.25\,y^9 + 0.98\,y^{10},$$
$$g_3 = 1.11169 + 4.05728\,y + 21.2208\,y^2 - 27.7251\,y^3 - 23.8725\,y^4 + 16.9743\,y^5 + $$
$$+ 14.0365\,y^6 - 1.98041\,y^7 - 3.93753\,y^8 - 0.536027\,y^9,$$
$$g_4 = 15.437 + 53.3631\,y + 287.127\,y^2 - 426.092\,y^3 - 274.673\,y^4 + 282.865\,y^5 + $$
$$+ 155.229\,y^6 - 54.8164\,y^7 - 46.3148\,y^8,$$
$$g_5 = -0.0107472 - 0.0728328\,y - 0.125708\,y2 + 0.659926\,y^3 - 0.648162\,y^4 + 0.0203181\,y^5 + $$
$$+ 0.307983\,y^6 - 0.132465\,y^7,$$
$$g_6 = -28.6209 - 146.467\,y - 466.802\,y^2 + 1191.69\,y^3 - 289.711\,y^4 - 484.562\,y^5 + 229.686\,y^6,$$
$$g_7 = 0.00719286 + 0.0711498\,y + 0.152208\,y^2 - 0.242717\,y^3 - 0.0750924\,y^4 + 0.0865886\,y^5 \;,$$
$$g_8 = 4.9153 - 68.943\,y + 153.899\,y^2 + 11.987\,y^3 - 106.64\,y^4,$$
$$g_9 = -0.00418028 - 0.117396\,y - 0.00190424\,y2 + 0.125103\,y3,$$
$$g_{10} = -5.26161 + 62.781\,y - 53.9861\,y^2,$$
$$g_{11} = 0.0181738 - 0.037381\,y$$
$$g_{12} = -12.5005$$

| | $g_0 = g$ | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ | $g_7$ | $g_8$ | $g_9$ | $g_{10}$ | $g_{11}$ | $g_{12}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $-18.5886$ | + | - | + | + | - | + | + | - | - | - | - | + | - | 7 |
| 18.5886 | + | + | + | - | - | - | + | + | - | + | - | - | - | 5 |
| | | | | | | | | | | | | | | **2** |

We can see that there are two real solutions to the equation.

As a result,

$$x_1 = -0.9798840171550919,\;\; x_2 = -0.09607531847629341$$
$$y_1 = -0.01076953317967882,\;\; y_2 = 0.9401726870760027.$$

This means that the four periods of a given mapping have four equal points.

$$(x_1, y_1) = (-0.9798840171550919,\; -0.01076953317967882),$$
$$(x_1, y_2) = (-0.9798840171550919,\, 0.9401726870760027),$$
$$(x_2, y_1) = (-0.09607531847629341,\; -0.01076953317967882),$$
$$(x_2, y_2) = (-0.09607531847629341,\, 0.9401726870760027).$$

At these points, we examine the spectra of the mapping given. That is, we find the modulus of the values of the equations in a given system of equations at points $x$ and $y$, respectively. It follows that this period is attractive because the absolute values of multiplier smaller then one.

Discussion.

For our mapping (2).

$$F_{c_1c_2}:\begin{cases} x' = y^2 + c_1, \\ y' = x^2 + c_2. \end{cases} \quad (4)$$

If $c_1 = c_2 = -1$ then all point of out site the rectangle $|x| \le \frac{1}{2}(1+\sqrt{5})$, $|y| \le \frac{1}{2}(1+\sqrt{5})$ tend to infinity. Some inside points tend to fixed points $(-1,0)$ *or* $(0,-1)$. And some inside points tend to periodic points with period two $(0,0)$ *and* $(-1,-1)$.

For example $x_0 = 0.776$, $y_0 = -0.36$.

| $n$ | $x_n$ | $y_n$ |
|---|---|---|
| $n=1$ | -0,8704 | -0,397824 |
| $n=2$ | -0,841736065024 | -0,24240384 |
| $n=3$ | -0,941240378353254 | -0,291480396837912 |
| ... | ... | ... |
| $n=10$ | -0,999970282135738 | -3,57147346333631E-6 |
| $n=11$ | -0,999999999987245 | -5,94348453725036E-5 |
| $n=12$ | 0,999999996467499 | -2,551092670E-11 |
| $n=13$ | -1 | -7,0650016823E-9 |
| $n=14$ | -1 | 0 |

For (32) when $c_1 = c_2 = -1$ then filled Julia set Fig. 2.3.1.

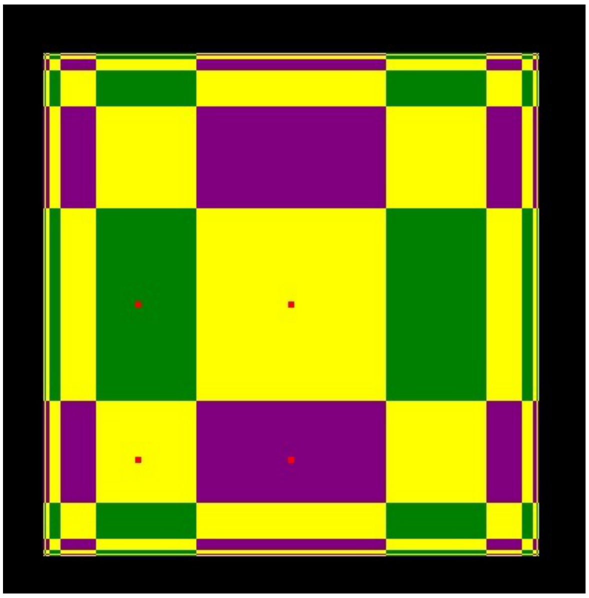

**Figure 1.** For $c_1 = c_2 = -1$ the classification of Julia set.

Let $c_1 = -0.98,\ c_2 = -0.02$ then all point in Julia set tend to the periodic points with period four.

For example $x_0 = 0.06,\ y_0 = -0.36$.

| $n$ | $x_n$ | $y_n$ |
|---|---|---|
| $n=1$ | -0,97973104 | 0,70318016 |
| ... | ... | ... |
| $n=16$ | 0,0960753164810756 | -0,0107895746542244 |
| $n=17$ | -0,9798835850787801 | -0,0107695335630612 |
| $n=18$ | -0,979884017146834 | 0,940171840306845 |
| $n=19$ | -0,0960769106940412 | 0,940172687059817 |
| $n=20$ | -0,0960753185067235 | -0,0107692272314892 |

Let $c_1 = -1.22,\ c_2 = -0.38$ then all point in Julia set tend to the periodic points with period eight but there are two cyclical points with period eight.

Classification all Cauchy problems for $c_1 = -1.22,\ c_2 = -0.38$ on the Figure 2.3.2.

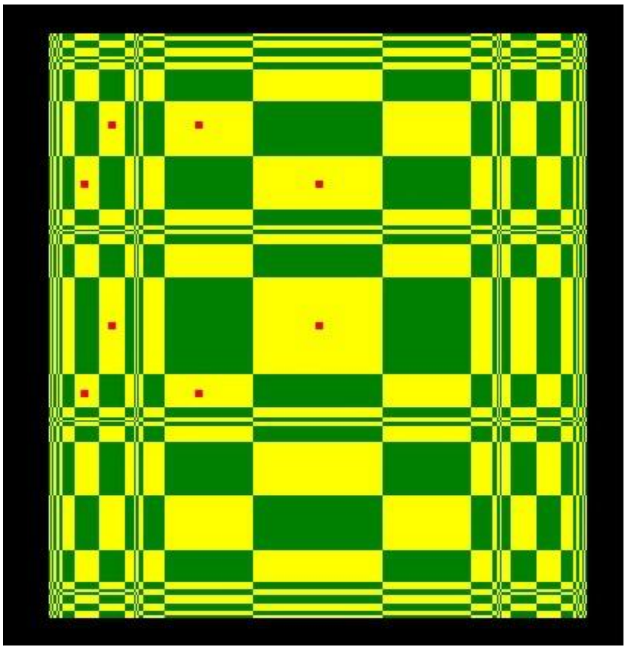

**Figure 2.** For $c_1 = -1.22,\ c_2 = -0.38$ the classification of Julia set.

Classification all Cauchy problems for $c_1 = -1.19,\ c_2 = -0.44.$ period 16, we get Fig. 2.3.3.

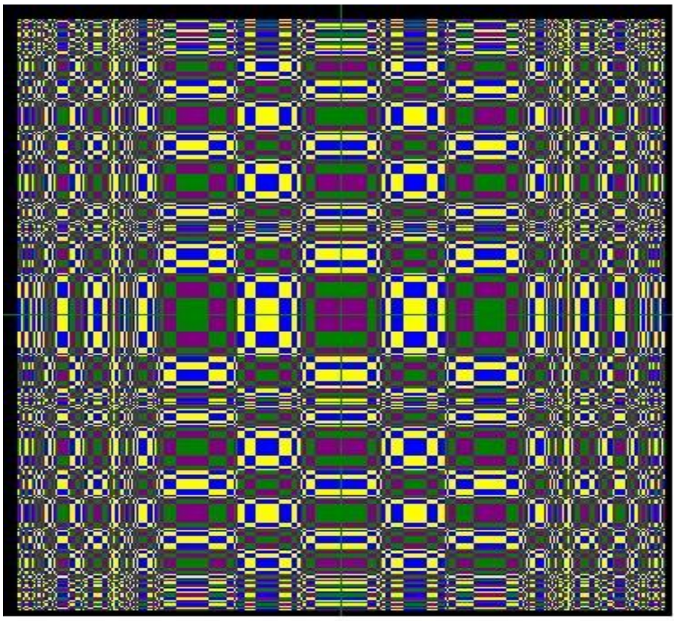

**Figure 3.** Classification all Cauchy problems for $c_1 = -1.19,\ c_2 = -0.44$ Period 3 and 6.

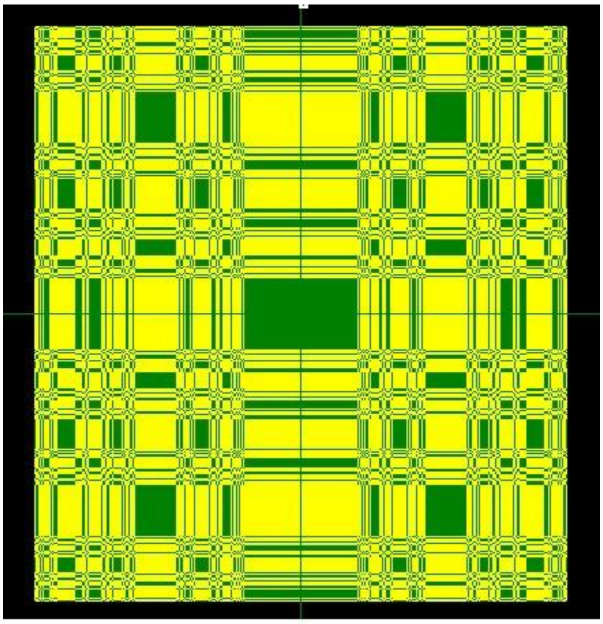

**Figure 4.** *For* $c_1 = -1.19,\ c_2 = -0.44$ *the classification of Julia set*

Classification all Cauchy problems for $c_1 = -1.31,\ c_2 = -0.8.$ Strange attractor.

## Conclusion

In conclusion, our study contributes to the broader understanding of quadratic mappings and their applications in ecological modeling. By elucidating the dynamics of periodic points, we pave the way for further research into the stability and resilience of population systems in complex environments. As we move forward, future research may explore additional dimensions of the mapping and consider more sophisticated numerical techniques for analyzing periodic orbits. Furthermore, the application of our findings in ecological modeling may lead to practical insights for managing and conserving natural ecosystems. Overall, our study represents a step towards unraveling the complexities of dynamical systems in ecological settings, offering new avenues for exploration and application in population ecology and related fields.

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